III. "Fundamental Views regarding Mechanics." By Professor Julius Plücker, of Bonn, For. Mem. R.S. Received May 29, 1866.

(Abstract.)

Being encouraged by the friendly interest expressed by English geometricians, I have resumed my former researches, which had been entirely abandoned by me since 1846. While the details had escaped from my memory, two leading questions have remained dormant in my mind. The first question was to introduce right lines as elements of space, instead of points and planes, hitherto employed; the second question, to connect, in mechanics, both translatory and rotatory movements by a principle in geometry analogous to that of reciprocity. I proposed a solution of the first question in the geometrical paper presented to the Royal Society. I met a solution of the second question, which in vain I sought for in Poinsot's ingenious theory of coupled forces, by pursuing the geometrical way. The indications regarding complexes of forces, given at the end of the "Additional Notes," involves it. I now take the liberty of presenting a new paper, intended to give to these indications the developments they demand, reserving for another communication a succinct abstract of the curious properties of complexes of right lines represented by equations of the *second* degree, and the simple analytical way of deriving them.

1. We usually represent a force geometrically by a limited line, *i. e.* by means of two points (x', y', z') and (x, y, z), one of which (x', y', z') is the point acted upon by the force, while the right line passing through both points indicates its direction, and the distance between the points its intensity. We may regard the six quantities

$$x-x', \quad y-y', \quad z-z', \quad yz'-y'z, \quad zx'-z'x, \quad xy'-x'y \quad \dots . \quad (1)$$

as the *six coordinates of the force.* The six coordinates of a force represent its three projections on the three axes of coordinates OX, OY, OZ, and its three moments with regard to the same axes. Accordingly we may, as far as we do not regard the point acted upon by the force, replace its coordinates by

$$X, \quad Y, \quad Z, \quad L, \quad M, \quad N \quad \dots\dots\dots\dots\dots \quad (2)$$

in admitting the equation of condition

$$LX+MY+NZ=0, \dots\dots\dots\dots\dots\dots\dots \quad (3)$$

which indicates that the axis of the resulting moment is perpendicular to the direction of the force. In making use of the primitive coordinates this condition is involved in the form given to them.

When the last condition is not satisfied, the coordinates X, Y, Z, L, M, N represent no longer a mere force; they are the coordinates of what I proposed to call a *dyname.*

In the general case such a dyname results when any numbers of given forces act upon any given points. Here the six coordinates of the dyname

are the sums of the six coordinates of the given forces (x', y', z', x, y, z). If between the six sums thus obtained,

$$\left.\begin{array}{lll}\Sigma(x-x'), & \Sigma(y-y'), & \Sigma(z-z'), \\ \Sigma(yz'-y'z), & \Sigma(zx'-z'x), & \Sigma(xy'-x'y),\end{array}\right\} \ldots\ldots\ldots \quad (4)$$

an equation analogous to (3) takes place, there is a resulting force. In the case of equilibrium the six sums become equal to zero.

2. In quite an analogous way as we have determined an ordinary force by means of two points in space, one of which is the point acted upon, we may represent a rotation, or the *rotatory force* producing it, by means of two planes,

$$t'x+u'y+v'z=1, \quad tx+uy+vz=1,$$

the coordinates of which are t', u', v' and t, u, v, one of the two planes (t', u', v') being the plane acted upon. The right line along which both planes meet is the axis of rotation. The plane acted upon may in a double way turn round the axis of rotation in order to coincide with the second plane (t, u, v); but there is no ambiguity in admitting that during the rotation the rotating plane does not pass through the origin, and consequently its coordinates do not become infinite.

Let us regard the six quantities

$$t-t', \quad u-u', \quad v-v', \quad uv'-u'v, \quad vt'-v't, \quad tu'-t'u \ \ldots\ldots \ (5)$$

as the six coordinates of the rotatory force. As far as we do not regard the plane acted upon, we may replace them by

$$\mathfrak{X}, \quad \mathfrak{Y}, \quad \mathfrak{Z}, \quad \mathfrak{L}, \quad \mathfrak{M}, \quad \mathfrak{N} \ \ldots\ldots\ldots\ldots\ldots \ (6)$$

in admitting the equation of condition,

$$\mathfrak{L}\mathfrak{X}+\mathfrak{M}\mathfrak{Y}+\mathfrak{N}\mathfrak{Z}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (7)$$

(For the geometrical signification of these new coordinates I refer to the original paper.)

When the last equation of condition is not satisfied, the coordinates $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ no longer represent a mere rotatory force; they are the six independent coordinates of what I called a *rotatory dyname*.

In the general case, such a rotatory dyname results when any number of given rotatory forces acts on any given planes. Here the six coordinates of the resulting rotatory dyname are the six sums of the six coordinates of the given rotatory forces.

$$\left.\begin{array}{lll}\Sigma(t-t'), & \Sigma(u-u'), & \Sigma(v-v'), \\ \Sigma(uv'-u'v), & \Sigma(vt'-v't), & \Sigma(tu'-t'u).\end{array}\right\} \ldots\ldots\ldots \quad (8)$$

If between these six coordinates the equation of condition subsists, there is a resulting rotatory force. In the case of equilibrium the six sums become equal to zero.

Any movement whatever may be referred as well to ordinary as to rotatory forces; consequently an ordinary dyname and a rotatory dyname mean the same.

3. From the notions developed we immediately obtain two general

theorems constituting the base of statics. In a similar way as d'Alembert's principle is derived from the "principe des vitesses virtuelles," both theorems may be transformed into fundamental theorems of mechanics.

In starting from the coordinates of ordinary forces, the equations of equilibrium are

$$\left.\begin{array}{lll}\Sigma(x-x')\quad=0, & \Sigma(y-y')\quad=0, & \Sigma(z-z')\quad=0, \\ \Sigma(yz'-y'z)=0, & \Sigma(zx'-z'x)=0, & \Sigma(xy'-x'y)=0,\end{array}\right\}\ldots\quad(9)$$

which, by replacing these forces by rotating ones, become

$$\left.\begin{array}{lll}\Sigma(t-t')\quad=0, & \Sigma(u-u')\quad=0, & \Sigma(v-v')\quad=0, \\ \Sigma(uv'-u'v)=0, & \Sigma(vt'-v't)=0, & \Sigma(tu'-t'u)=0.\end{array}\right\}\ldots\quad(10)$$

We likewise may express the conditions of equilibrium by the following six equations—

$$\left.\begin{array}{lll}\Sigma(x-x')=0, & \Sigma(y-y')=0, & \Sigma(z-z')=0, \\ \Sigma(t-t')=0, & \Sigma(u-u')=0, & \Sigma(v-v')=0,\end{array}\right\}\ldots\ldots\quad(11)$$

three of which are taken amongst (9), three amongst (10)—and expand these equations in the analytical way, starting either from the consideration of ordinary or rotatory forces. The interpretation of these equations immediately gives the following two theorems. In the case of equilibrium—

I. *The centre of gravity of the points acted upon by the forces coincides with the centre of gravity of the second points, by means of which the forces are determined* (No. 1).

II. *The central plane of the planes acted upon by the rotatory forces coincides with the central plane of the second planes, by means of which the rotatory forces are determined* (No. 2)*.

If we introduce the notion of *masses*, both theorems hold good, only the definition of both kinds of forces and therefore their unity is changed. The points acted upon become centres of gravity corresponding to masses; the planes acted upon, central planes corresponding to moments of inertia.

If equilibrium does not exist, we get, in the general case, one resulting force, determined by the two centres of gravity, and one resulting rotatory force determined by the two central planes. We easily obtain the six differential equations of the movement produced.

I shall think it suitable further to develope the principles merely indicated in the paper presented. A Treatise on Mechanics, reconstructed on them, will assume quite a new aspect.

4. In making use, within a plane, of point- or line-coordinates, we represent, by means of an equation between the two coordinates, a plane curve described by a point, or enveloped by a right line. In making use, in space, of point- or plane-coordinates, by means of an equation between the three coordinates, a surface is represented which may be regarded as a

* The coordinates of the central plane are obtained in the same way by means of the coordinates of the given planes as the coordinates of the centre of gravity are obtained by means of the coordinates of the given points.

locus of points, or an envelope by planes. In my geometrical paper I introduced the right line, depending on four constants, as the element of space. An equation between these constants, if regarded as variables, represents a *complex of lines*. Each line of the complex may be regarded either as a ray described by a point, or as an *axis* enveloped by planes. In order to render the developments symmetrical, I adopted as coordinates of the right line the same expressions (1) and (4) (connected by an equation of condition) made use of in the determination of ordinary and rotatory forces. At the same time, general equations between the six coordinates were replaced by homogeneous ones. In doing so, A, B, C, D, E, F denoting any six constants,

$$\left.\begin{array}{l}\mathrm{A}(x-x')+\mathrm{B}(y-y')+\mathrm{C}(z-z')+\mathrm{D}(yz'-y'z)\\ \quad+\mathrm{E}(yx'-y'x)+\mathrm{F}(xy'+x'y')\equiv\Omega=0,\end{array}\right\}\ldots\ldots\ldots\ldots (12)$$

$$\left.\begin{array}{l}\mathrm{A}(uv'-u'v)+\mathrm{B}(vt'-v't)+\mathrm{C}(tu'-t'u)\\ \quad+\mathrm{D}(t-t')+\mathrm{E}(u-u')+\mathrm{F}(v-v')\equiv\Phi=0\end{array}\right\}\ldots\ldots\ldots\ldots (13)$$

represent the *same* linear complex of lines, which may be regarded as a complex of rays as well as a complex of axes.

When general equations between the same six coordinates are admitted, the complex of rays becomes a *complex of ordinary forces*, the complex of axes a *complex of rotations* or *rotatory forces*, both ordinary and rotatory forces depending upon five constants. Here we meet a reciprocity between both kind of forces corresponding to the reciprocity between point and plane.

Finally, in omitting the equation of condition hitherto made use of, we pass from forces to dynames, depending upon the six independent constants

$$\mathrm{X},\ \ \mathrm{Y},\ \ \mathrm{Z},\ \ \mathrm{L},\ \ \mathrm{M},\ \ \mathrm{N},\ldots\ldots\ldots\ldots (14)$$

or

$$\mathfrak{X},\ \ \mathfrak{Y},\ \ \mathfrak{Z},\ \ \mathfrak{L},\ \ \mathfrak{M},\ \ \mathfrak{N}.\ldots\ldots\ldots\ldots (15)$$

A complex of dynames is represented by an equation between six variables. Here again, as in the case of right lines, the same complex may be represented in a double way by means of the two sets of coordinates.

In order to complete these general considerations, we add the following. A dyname may be resolved into *two* variable forces, either ordinary or rotatory. These variable forces constitute a linear complex, either of ordinary or extraordinary forces. A homogeneous equation between the six *independent* coordinates (14) or (15) represents a *complex of two coupled variable right lines*.

5. I will conclude by giving some details regarding complexes of the first degree.

A linear complex of ordinary forces is represented by the linear equation

$$\Omega=1,$$

which may be expanded thus:

$$\left.\begin{array}{l}(\mathrm{A}+\mathrm{F}y'-\mathrm{E}z')x\\ +\ (\mathrm{B}-\mathrm{F}x'-\mathrm{D}z')y\\ +\ (\mathrm{C}+\mathrm{E}x'-\mathrm{D}y')z\\ =\mathrm{A}x'+\mathrm{B}y'+\mathrm{C}z'.\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

In putting x', y', z' as constants, those forces of the complex are obtained which pass through the given point (x', y', z'). In this supposition the equation of the complex becomes the equation of a plane. This plane is the locus of the second points, by means of which the forces of the complex are determined. Hence in a linear complex there are acting on each point of space forces in all directions, the intensity of each force being the segment on its direction between the point acted upon and its corresponding plane (16).

Hence we derive the following theorem :—

In a complex of rotatory forces any given plane of space is acted upon by an infinite number of forces, each line within the plane being an axis of rotation. The rotations round all axes are determined by second planes passing through a fixed point, the position of which depends upon the given point.

I showed in the geometrical paper the way of discussing linear complexes of right lines. The properties of linear complexes of forces, either ordinary or rotatory, may be developed in a similar way.

6. Right lines belonging simultaneously to two complexes constitute a single congruency, and accordingly intersect two fixed lines; if belonging to three complexes, they constitute a double congruency, *i. e.* one generation of a hyperbolic or parabolic hyperboloid. Forces, either ordinary or rotatory, belonging simultaneously to two, three, four complexes, constitute a single, double, or threefold congruency of forces. In admitting these denominations, the following results are immediately obtained.

In a linear congruency of ordinary or rotatory forces, the directions of all ordinary, or the axes of all rotatory forces, constitute a linear complex of lines. All ordinary forces acting on any given point of space are confined within the same plane; the intensity of each force is equal to the distance between the point acted upon and the point where its direction meets a fixed line within the mentioned plane. The axes of all rotatory forces, acting upon any given plane of space, meet in a fixed point within that plane. There is a fixed line passing through the fixed point, round which the second planes of all forces turn when their axes turn round the fixed point in the given plane acted upon.

In a double congruency of ordinary forces there is one force of given direction and intensity acting upon any point of space, as there is in a double congruency of rotatory forces one force acting upon any given plane.

In a threefold congruency of ordinary forces the *directions*, in a threefold congruency of rotatory forces the *axes* of all forces constitute one of the two generations of a hyperboloid.

These few indications may be sufficient here; but before concluding I must, in referring to the original paper, make a last remark. Forces acting along a given right line may either be regarded as the same, whatever may be the point of the line acted upon, or they may be regarded as varying in intensity according to the position of the point. There is an analogous distinction to be made with regard to rotatory forces. Accordingly two different kinds of complexes must be distinguished.